ITERATED INTEGRAL AND KNIZHNIK–ZAMOLODCHIKOV EQUATIONS

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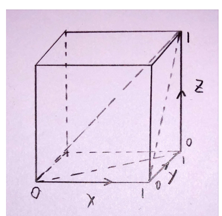
1. Introduction

An iterated integral, which has a nested structure, looks like

$$\int_0^1 \left(\int_0^z \left(\int_0^y f(x,y,z)dx\right) dy\right) dz, \text{or simply} \int_0^1 \int_0^z \int_0^y f(x,y,z)dxdydz,$$

where $0 \leq x \leq y \leq z \leq 1$. The integration domain is a simplex in the cube $[0,1]^3$, the usual domain of integration when x, y, z are not ordered.



Although iterated integral is a basic technique in calculus, the theory of iterated integral invented by K. T. Chen ([4]) was for the purpose of constructing functions on the infinite-dimensional space of paths on a manifold ([3]) and has been seen a lot of connections to modern mathematics and physics. In this thesis, a brief introduction to iterated integrals and some of its properties is first introduced, then through interrelated examples in differential equations, knot thoery, stochastic processes ... we try to reveal some common characteristics, thus we can hope to apply techniques in one area to another through the theme of iterated integrals.

2. Iterated Integrals

Definition 1. *Let M be a real or complex manifold and ω_1, ..., ω_n be 1-forms on M and let $\gamma : [0,1] \to M$ be a smooth path. Write*

$$\gamma^* \omega_i = f_i(t)dt, \tag{1}$$

for the pullback of the forms ω_i *to the interval* $[0,1]$ *and define*

$$\begin{aligned}\int_\gamma \omega_1 \dots \omega_n &= \int_{0\leq t_1\leq \dots \leq t_n\leq 1} f_n(t_n)\cdots f_1(t_1)dt_1\dots dt_n \\ &= \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_n)f_{n-1}(t_{n-1})\dots f_1(t_1)dt_1dt_2\dots dt_n,\end{aligned} \tag{2}$$

which will be called the iterated integral of ω_1, ..., ω_n *along* γ. *Moreover, define the iterated integral of empty product of 1-forms to be* 1.

Proposition 1. *Iterated integrals satisfy the following properties:*

(a) If $\gamma^{-1}(t) = \gamma(1-t)$ *is the inverse path of the path* γ, *then*

$$\int_{\gamma^{-1}} \omega_1 \dots \omega_n = (-1)^n \int_\gamma \omega_n \dots \omega_1. \tag{3}$$

(b) (The composition property) If α, β *are two paths, denote the path obtianed by first traversing* β *then* α *by* $\alpha\beta$, *then*

$$\int_{\alpha\beta} \omega_1 \dots \omega_n = \sum_{i=0}^{n} \int_\alpha \omega_1 \dots \omega_i \int_\beta \omega_{i+1} \dots \omega_n. \tag{4}$$

(c) The shuffle property:

$$\int_\gamma \omega_1 \dots \omega_r \int_\gamma \omega_{r+1} \dots \omega_{r+s} = \sum_{\sigma \in \mathrm{Shuff}(r,s)} \int_\gamma \omega_{\sigma(1)} \dots \omega_{\sigma(r+s)}, \tag{5}$$

where $\mathrm{Shuff}(r,s)$ *is the set:*

$$\mathrm{Shuff}(r,s) = \begin{Bmatrix} \textit{all permutations } \sigma \textit{ of } (r+s) \textit{ indices such that} \\ \sigma(1) < \dots < \sigma(r) \textit{ and } \sigma(r+1) < \dots < \sigma(r+s) \end{Bmatrix}.$$

Proof. (a)

$$\begin{aligned}\int_{\gamma^{-1}} \omega_1 \dots \omega_n &= \int_1^0 \int_1^{t_1} \cdots \int_1^{t_{n-1}} f_1(t_1)f_2(t_2)\dots f_n(t_n)dt_ndt_{n-1}\dots dt_1 \\ &= \int_1^0 \int_{t_n}^0 \cdots \int_{t_2}^0 f_n(t_n)f_{n-1}(t_{n-1})\dots f_1(t_1)dt_1dt_2\dots dt_n \\ &= (-1)^n \int_\gamma \omega_n \dots \omega_1.\end{aligned} \tag{6}$$

.

(b) Let $0 < x < 1$, then

$$\begin{aligned}&\{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n : 0 \le t_1 \le t_2 \le \cdots \le t_n \le 1\}\\ &= \bigcup_{k=0}^{n} \{(t_1, \dots, t_k) \in \mathbb{R}^k : 0 \le t_1 \le \ldots \le t_k \le x\} \\ &\quad \times \{(t_{k+1}, \dots, t_n) \in \mathbb{R}^{n-k} : x \le t_{k+1} \le \ldots \le t_{n-1} \le_{t_n}\}.\end{aligned} \tag{7}$$

(c) On the R.H.S. of (5), all $(r+s)$ indices are ordered, but on the L.H.S. only the r indices and the s indices are ordered, so on the R.H.S. we need to count all permutations such that only the r indices and the s indices are ordered. □

3. Dyson Series

In this section, we introduce an example of iterated integrals as solutions to differential equations.

3.1. **Schrödinger Equation for the Time-evolution Operator.** Let time t be a parameter that ranges over $\mathbb{R}$, a state ket $|v, t\rangle$ denote a vector $\mathbf{v}$ in a complex vector space V representing the state of some quantum system at time t. We would like to study the time evolution of the state ket. Suppose at time t_0 the state ket of the system is $|\alpha, t_0\rangle$, at some later time $t > t_0$, let us denote the state ket of the system by $|\alpha, t_0; t\rangle$. Define the time-evolution operator $\mathcal{U}(t, t_0)$ by an equation which relates the two kets:

$$|\alpha, t_0; t\rangle = \mathcal{U}(t, t_0)|\alpha, t_0\rangle.$$

Proposition 2. *The time-evolution operator satisfies the following properties* ([2])*:*

(a) $\mathcal{U}^\dagger(t, t_0)\mathcal{U}(t, t_0) = 1.$ *(conservation of probability)*

(b) $\mathcal{U}(t_2, t_0) = \mathcal{U}(t_2, t_1)\mathcal{U}(t_1, t_0)$, $(t_2 > t_1 > t_0)$. *(the compostion property)*

Now we would like to consider an infinitesimal time-evolution operator $\mathcal{U}(t_0 + dt, t_0)$. If we take

$$\mathcal{U}(t_0 + dt, t_0) = 1 - i\Omega dt, \tag{8}$$

where Ω is a Hermitean operator,

$$\Omega^\dagger = \Omega,$$

it can satisfy the two properties for the time-evolution operator. From classical mechanics we know that the Hamiltonian generates the time evolution, so we should relate Ω to the Hamiltonian H. Since the dimension of Ω is inverse of time, by Planck-Einstein relation,

$$E = \hbar\omega, \tag{9}$$

the relation should naturally be

$$H = \hbar\Omega, \tag{10}$$

where H is assumed to be Hermitian. Therefore, we take the form of the infinitesimal time-evolution operator to be

$$\mathcal{U}(t_0 + dt, t_0) = 1 - \frac{iHdt}{\hbar}. \tag{11}$$

Applying the composition property to the time evolution operator at t_0, $t_1 = t$ and $t_2 = t + dt$ we have

$$\begin{aligned} \mathcal{U}(t + dt, t_0) &= \mathcal{U}(t + dt, t)\mathcal{U}(t, t_0) \\ &= (1 - \frac{iHdt}{\hbar})\mathcal{U}(t, t_0), \end{aligned} \tag{12}$$

hence

$$\mathcal{U}(t + dt, t_0) - \mathcal{U}(t, t_0) = -\frac{iHdt}{\hbar}\mathcal{U}(t, t_0). \tag{13}$$

Combining Taylor series of $\mathcal{U}(t + dt, t_0)$ to the first derivative

$$\mathcal{U}(t + dt, t_0) = \mathcal{U}(t, t_0) + \frac{\partial U(t, t_0)}{\partial t}dt \tag{14}$$

gives us the Schrödinger equation for the time-evolution operator

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}(t, t_0) = H\mathcal{U}(t, t_0). \tag{15}$$

Now let us solve this equation. In general, the Hamiltonians are time-dependent and do not commute at differet times. Note that by the definition of the time-evolution operator,

$$\mathcal{U}(t_0, t_0) = 1, \tag{16}$$

and (8) is equivalent to the integral equation

$$\mathcal{U}(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H(s)\mathcal{U}(s, t_0)ds, \tag{17}$$

we can apply Picard's method to solve this equation by successive approximation ([3]): let $\mathcal{U}_0(t, t_0) = \mathcal{U}(t_0, t_0) = 1$ be the constant function and define for $n \geq 0$,

$$\mathcal{U}_{n+1}(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H(s)\mathcal{U}_n(s, t_0)ds, \tag{18}$$

then assuming $t_0 \leq t_1 \leq \ldots \leq t_n = t$,

$$\begin{aligned}
\mathcal{U}_1(t,t_0) &= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H(t_1)\mathcal{U}_0(t,t_0)dt_1,\\
\mathcal{U}_2(t,t_0) &= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H(t_2)\mathcal{U}_0(t,t_0)dt_1\\
&\quad + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t}\int_{t_0}^{t_2} H(t_2)H(t_1)\mathcal{U}_1(t,t_0)\mathcal{U}_2(t_1,t_0)dt_1dt,\\
&\ldots \qquad (19)\\
\mathcal{U}_n(t,t_0) &= 1 + \sum_{n\geq 1}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t}\int_{t_0}^{t_{n-1}}\cdots\int_{t_0}^{t_2} H(t)\ldots H(t_2)H(t_1)\mathcal{U}_0(t,t_0)dt_1dt_2\ldots dt,\\
&= 1 + \sum_{n\geq 1}\left(\frac{-i}{\hbar}\right)^n \int_{t_0\leq t_1\leq t_2\leq\ldots\leq t} H(t)\ldots H(t_2)H(t_1)dt_1dt_2\ldots dt,\\
&\ldots
\end{aligned}$$

If $\lim_{n\to\infty}\mathcal{U}_n(t,t_0)$ exists, it gives the solution to (15) and is called the Dyson series. The infinite sum is an example of iterated integrals. In physics literatures, the Dyson series is written as

$$\mathcal{U}(t,t_0) = \mathcal{T}\left(\exp\left(\frac{-i}{\hbar}\int_{[t_0,t]} H(t)dt\right)\right), \qquad (20)$$

where $\mathcal{T}$ is the time-ordering operator.

4. The KZ equation

Let us now see another example in which iterated integrals appear as solutions to differential equations. First, we introduce the concept of multiple polylogarithms.

Definition 2. *Multiple polylogarithms are nested sum of the form* ([6])

$$\mathrm{Li}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) := \sum_{n_1>\ldots>n_k>0}\prod_{j=1}^{k}\frac{z_j^{n_j}}{n_j^{s_j}},$$

where $s_1,\ldots,s_k$ and $z_1,\ldots,z_k$ are complex numbers such that the sum converges. In particular, setting $z_j = 1$ gives us multiple zeta values

$$\zeta(s_1,\ldots,s_k) := \sum_{n_1>\ldots>n_k>0}\prod_{j=1}^{k}\frac{1}{n_j^{s_j}}$$

We now define a specific type of multiple polylogarithms which is relevant to the KZ equation. Let $X = \{x_0, x_1\}$ be an alphabet in two letters and $X^\times$ be the set of words in x_0, x_1 and the empty word e. Let $\mathbb{Q}\langle x_0, x_1\rangle$ be the vector space generated by the words in X, equipped with the shuffle product:

$$x_{i_1}\dots x_{i_r} \sqcup\!\sqcup\, x_{i_{r+1}}\dots x_{i_{r+s}} = \sum_{\sigma\in \mathrm{Shuff}(r,s)} x_{\sigma(1)}\dots x_{\sigma(r+s)}, \tag{21}$$

and where $e \sqcup\!\sqcup\, w = w \sqcup\!\sqcup\, e = w$ for all $w \in X^*$. To every non-empty word $w \in X^*$, we associate a multivalued function $\mathrm{Li}_w(z)$ as follows:

1) If w ends with x_1, write $w = x_0^{n_1-1}x_1 \dots x_0^{n_k-1}x_1$ and let

$$\mathrm{Li}_w(z) = \int_0^z \omega_1\omega_0^{n_1-1}\omega_1\dots\omega_0^{n_k-1} = \mathrm{Li}_{n_1,\dots,n_k}(z) \tag{22}$$

where $\omega_0 = \frac{dz}{z}$, $\omega_1 = \frac{dz}{1-z}$ are 1-forms defined on $M = \mathbb{C}\setminus\{0,1\}$ and $z_1, \dots, z_{k-1}$ are set to 1. To see the equality, note that

$$\frac{d}{dz}\mathrm{Li}_{n_1,\dots,n_k}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}_{n_1,\dots,n_k-1}(z), & \text{if} \quad n_k > 1, \\ \frac{1}{1-z}\mathrm{Li}_{n_1,\dots,n_{k-1}}(z), & \text{if} \quad n_k = 1. \end{cases} \tag{23}$$

It follows that for $n_k > 1$,

$$\mathrm{Li}_{n_1,\dots,n_k}(z) = \int_\gamma \mathrm{Li}_{n_1,\dots,n_k-1}(z)\frac{dt}{t}, \tag{24}$$

then let $0 \le \dots \le t'_1 \le \dots \le t'_{n_2} \le t_1 \le \dots \le t_{n_1} \le 1$,

$$\begin{aligned}\int_\gamma \omega_1\omega_0^{n_1-1}\omega_1\dots\omega_0^{n_k-1} &= \dots\left(\int_{0\le t'_1\le\dots\le t'_{n_2}\le 1}\left(\int_{0\le t_1\le\dots\le t_{n_1}\le 1}\frac{zdt_1}{1-zt_1}\frac{dt_2}{t_2}\cdots\frac{dt_{n_1}}{t_{n_1}}\right)\frac{zdt'_1}{1-zt'_1}\frac{dt'_2}{t'_2}\cdots\frac{dt_{n'_2}}{t'_{n_2}}\right)\dots \\ &= \mathrm{Li}_{n_1,\dots,n_k}(z).\end{aligned} \tag{25}$$

2) Set

$$\mathrm{Li}_{x_0^n}(z) = \frac{1}{n!}\log^n(z), \tag{26}$$

then any word can be composed by x_0^n and words which end with x_1 by the composition property of iterated integrals.

Consider the generating series

$$L(z) = \sum_{w\in X^*} w\mathrm{Li}_w(z), \tag{27}$$

it defines a multivalued function on M taking values in

$$\mathbb{C}[[X]] = \{\sum_{w\in X^*} S_w w : S_w \in \mathbb{C}\}, \tag{28}$$

the ring of non-commutative formal power series in the words X^*, with the multiplication being the concatenation of words ($w_1 \times w_2 = w_1 w_2$).

By (23), $L(z)$ satisfies the differential equation

$$\frac{d}{dz}L(z) = \left(\frac{x_0}{z} + \frac{x_1}{z-1}\right) L(z), \tag{29}$$

which is the one-dimensional Knizhnik–Zamolodchikov equation.

Note that for words $w \neq x_0^n$,

$$\lim_{z\to 0} \mathrm{Li}_w(z) = 0, \tag{30}$$

we have

$$L(z) \sim \exp(x_0 \log z) \quad \text{as} \quad z \to 0. \tag{31}$$

Similarly, there exists another solution

$$L^1(z) \sim \exp(x_1 \log(1-z)) \quad \text{as} \quad z \to 1. \tag{32}$$

Define the Drinfel'd associator $\Phi(z)$ by an equation which relates the two solutions:

$$L(z) = L^1(z)\Phi(z), \tag{33}$$

we denote it by $\Phi(x_0, x_1)$.

Proposition 3. $\Phi(x_0, x_1)$ *is a constant series.*

Proof. Differentiating $L^1(z)\Phi(z) = L(z)$ and use (29), we have

$$L^1 d\Phi(z) = 0. \tag{34}$$

Since L^1 is invertible, $d\Phi(z) = 0$. □

Proposition 4. *The coefficients of* $\Phi(x_0, x_1)$ *are multiple zeta values.*

Proof. By (32),

$$\Phi(x_0, x_1) = \lim_{z\to 1^-} (\exp(-x_1 \log(1-z))L(z)), \tag{35}$$

so for words $w \in x_0 X^* x_1$,

$$\Phi(x_0, x_1) = L(1) = \sum w\zeta(w). \tag{36}$$

Next, we can use the shuffle property to write any word by x_0, x_1 and $x_0X^*x_1$:

$$\begin{aligned}
&\zeta(x_0) = \zeta(x_1) = 0,\\
&\zeta(x_0x_0) = \frac{1}{2}\zeta(x_0 \sqcup\!\sqcup x_0) = \frac{1}{2}\zeta(x_0)\zeta(x_0) = 0,\\
&\zeta(x_0x_1) = \zeta(2),\\
&\zeta(x_1x_0) = \zeta(x_1 \sqcup\!\sqcup x_0 - x_0x_1) = \zeta(x_0)\zeta(x_1) - \zeta(x_0x_1) = -\zeta(2),\\
&\zeta(x_1x_1) = \frac{1}{2}\zeta(x_1 \sqcup\!\sqcup x_1) = 0,\\
&\dots
\end{aligned} \tag{37}$$

We have

$$\Phi(x_0, x_1) = 1 + \zeta(2)[x_0, x_1] + \zeta(3)([x_0, [x_0, x_1] - [x_0, x_1], x_1]) + \dots. \tag{38}$$

□

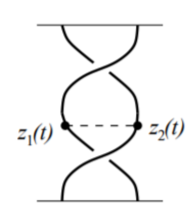
In the next section, we will interpret this formula in terms of knots.

5. The Kontsevich Integral

In this section, we show how iterated integrals appear in knot theory.

In the construction of the Kontsevich integral, $\mathbb{R}^3$ where the tangles are defined is represented as a product of complex plane $\mathbb{C}$ with coordinate z and real line $\mathbb{R}$ with coordinate t.

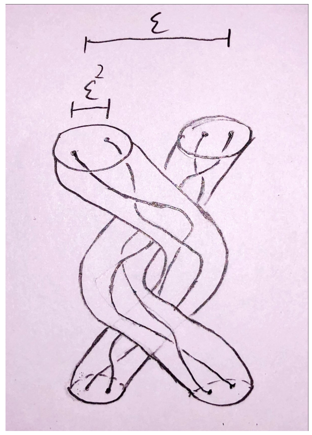
Intuitively, the Kontsevich integral counts the number of "twists" between strands in a tangle so it's always an integer or half an integer. For example, the number of twists between two strands in the right figure can be computed as

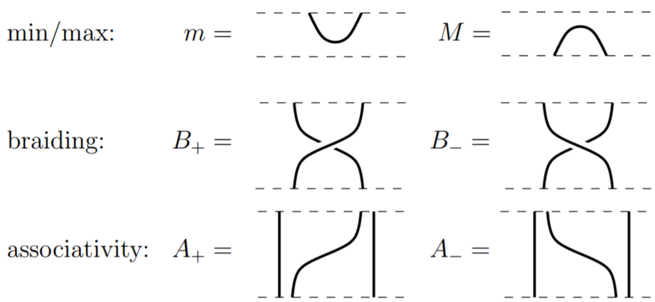


$$\frac{1}{2\pi i}\int_0^1 \frac{dz_1 - dz_2}{z_1 - z_2}.$$

The Kontsevich integral can be regarded as a way of counting twists at various "scales": there could be twists of twists, for example, in the following figure, the twists of tubes produce an underlying twist for the twisting strands inside the tubes, if we call the twists of tube "first order" (ε), then the twists of strands are of second order (ε^2).

To construct the Kontsevich integral, cut the knot into several parts by some slices on t such that each part contains one of the three basic events:

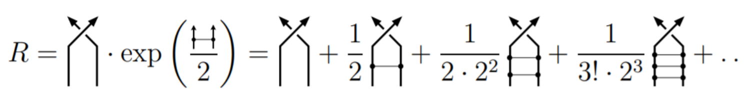




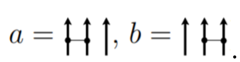
Now we can replace a knot by chord diagrams according to the followings rules:

$$
\begin{aligned}
&m, M \mapsto 1, \\
&B_+ \mapsto R, \quad B_- \mapsto R^{-1}, \\
&A_+ \mapsto \Phi, \quad A_- \mapsto \Phi^{-1}
\end{aligned}
\tag{39}
$$

where



$$\Phi = 1 - \frac{\zeta(2)}{(2\pi i)^2}[a,b] - \frac{\zeta(3)}{(2\pi i)^3}([a,[a,b]] + [b,[a,b]) + \dots$$



Then compute the product of these diagrams from top to bottom (see details in [8]).

Φ is the Drinfel'd associator in chord diagrams which relate three adjacent strands in a knot and chord diagrams in R relate two adjacent strands, so the Knotsevich integral gives us a way to obtain information of all twists by decomposing them into local information only involved in two or three strands. Also note that the more chords we use in the construction, the more information we get in higher order twists, which is similar to the construction of the Dyson series.

6. $\mathcal{T}$-exponential

In this section, we further discuss the $\mathcal{T}$-exponential we introduced in (20). We first introduce an alternative way to solve for (15): divide the interval $[0,t]$ into N parts evenly by $0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = t$ so $t_k = tk/N$. If we are given an initial condition at t_{k-1}, then the solution to (15) at t_k is $\mathcal{U}(t_k, t_{k-1})$.

Now let $N \to \infty$, then

$$\begin{aligned}
\mathcal{U}(t_1,0) &= 1 - \frac{iH(0)}{\hbar} \\
\mathcal{U}(t_2,0) &= \mathcal{U}(t_2,t_1)\mathcal{U}(t_1,0) = \left(1 - \frac{iH(t_1)}{\hbar}\right)\left(1 - \frac{iH(0)}{\hbar}\right) \\
&\cdots \\
\mathcal{U}(t,0) &= \lim_{N\to\infty}\left(1 - \frac{iH(t_{N-1})}{\hbar}\right)\cdots\left(1 - \frac{iH(t_1)}{\hbar}\right)\left(1 - \frac{iH(0)}{\hbar}\right) \\
&= \lim_{N\to\infty}\left(1 - \frac{iH(t(N-1)/N)}{\hbar}\right)\cdots\left(1 - \frac{iH(t/N)}{\hbar}\right)\left(1 - \frac{iH(0)}{\hbar}\right)
\end{aligned} \tag{40}$$

It is interesting to consider the Hamiltonians $H(t)$ as a random process and study the statistics of the time-evolution operator $\mathcal{U}(t,0) = \mathcal{T}\left(\exp\left(\frac{-i}{\hbar}\int_{[0,t]} H(t)dt\right)\right)$, the product form of the solution gives us an easier way to calculate them.

Let $H(t_k) \in \mathrm{Mat}(n \times n, \mathbb{C})$ be a stationary random process, its statistics is defined by the measure

$$P[H]DH = \lim_{N\to\infty}\prod_{k=0}^{N}\prod_{i,j=1}^{n} P[H(t_k)]dH_{ij}(t_k), \tag{41}$$

where $P[H(t)]$ is the probability density functional. We are interested in the average

$$\langle F[\mathcal{U}]\rangle = \int F[\mathcal{U}]P[H]DH, \tag{42}$$

where F is a functional of $\mathcal{U}(t,0)$.

Using the product form of the solution, we can perform the Iwasawa decomposition ([8]):

$$\mathcal{U}(t_k,0) = R(t_k)D(t_k)S(t_k), \tag{43}$$

where R, D, S represent "rotational" "diagonal" and "shear" degrees of freedom of $\mathcal{U}(t_k,0)$: R is an orthogonal matrix ($R_{ki}R_{kj} = \delta_{ij}$), D is diagonal, S is an an upper triangular matrix with diagonal elements equal to 1.

Substitue (43) into (15), we get

$$H = i\hbar\frac{\partial}{\partial t}(RDS)S^{-1}D^{-1}R^{-1} = R(R^T\partial_t R + (\partial_t D)D^{-1} + (\partial_t D)SS^{-1}D^{-1})R^T. \tag{44}$$

Denote $r = R^T\partial_t R$, $d = (\partial_t D)D^{-1}$, $s = D(\partial_t S)S^{-1}D^{-1}$, $X = r + d + s$, then

$$H = i\hbar RXR^T, \tag{45}$$

Proposition 5. ([8]) *d is diagonal, s is an upper triangular matrix with zeroes in the main diagonal and r is antisymmetric:*

$$d = \mathrm{diag}(d_1, \dots, d_n), \quad s_{ij} = 0 \text{ if } i > j, \quad r_{ij} = -r_{ji}. \tag{46}$$

Now consider $X(t)$ as an independent functional variable then (46) should be understood as

$$H[X] = i\hbar R[X]XR^T[X]. \tag{47}$$

Note that R only depends on r-component of X, we now can study the statistics of $\mathcal{T}$-exponential by a simple rotational $\mathcal{T}$-exponential:

$$\langle F[X]\rangle = i\hbar\int F[\mathcal{U}[X]]P[R[X]XR^T[X]]J[X]DX, \tag{48}$$

where $J[X]$ is the Jacobian. (See [9], [10] for concrete random processes)

Another statistics of interests is the Lyapunov spectrum:

Definition 3. *The Lyapunov spectrum $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of*

$$\Lambda = \lim_{t\to\infty}\frac{1}{t}\log(\mathcal{U}(t,0)), \tag{49}$$

so Iwasawa decompostion of $\mathcal{U}(t,0)$ also gives us an easier way to calculate them: they are the diagonal elements of $D(t)$.

These statistics, which is relevant to physics, will hopefully shed light on the zeta function.

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